# Triple Integrals

Defn: The triple integration of a function $f(x,y,z)$ defined over a closed parallelopiped, is expressed in the same manner as in the case of a double integral. Thus, we first obtain the sum

$$S = \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k)\, \delta x_i\, \delta y_j\, \delta z_k .$$

Then the triple integral of $f(x,y,z)$ over a region $V$ is the limiting value to which the sum will tend to as $n \to \infty$.

Thus, $$\iiint_V f(x,y,z)\, dv = \lim_{n \to \infty} \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k)\, \delta x_i\, \delta y_j\, \delta z_k$$

The integral $$\int_{x=a}^{b} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dx\, dy\, dz$$

is generally evaluated as

$$\int_{x=a}^{b} \left[ \int_{y=\phi_1(x)}^{\phi_2(x)} \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz \right\} dy \right] dx$$

or $$\int_{z=f_1(x,y)}^{f_2(x,y)} \left[ \int_{y=\phi_1(x)}^{\phi_2(x)} \left\{ \int_{x=a}^{b} f(x,y,z)\, dx \right\} dy \right] dz$$

(1) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$.

**Sol$^n$:** We can write the given integral as

$$I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y} \cdot e^{z}\, dz\, dy\, dx = \int_0^a \int_0^x e^{x+y} \left[e^{z}\right]_0^{x+y} dy\, dx$$

$$= \int_0^a \int_0^x e^{x+y}\left(e^{x+y} - 1\right) dy\, dx = \int_0^a \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^{x} \cdot e^{y}\right]_{y=0}^{x} dx$$

$$= \int_0^a \left[\frac{1}{2} e^{2x}\left(e^{2x} - 1\right) - e^{x}\left(e^{x} - 1\right)\right] dx$$

$$= \left[\frac{1}{8} e^{4x} - \frac{3}{4} e^{2x} + e^{x}\right]_0^a = \frac{1}{8} e^{4a} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}.$$

(2) Evaluate $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$. Ans

**Sol$^n$:** Integrating first w.r.t. $y$ keeping $x$ and $z$ const we have

$$I = \int_{-1}^{1} \int_0^z \left[xy + \frac{y^2}{2} + yz\right]_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1} \int_0^z \left[(x+z)(2z) + \frac{1}{2} 4xz\right] dx\, dz$$

$$= 2\int_{-1}^{1} \left[\frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z\right]_0^z dz = 2\int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right) dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 0$$

Ans

(b) Cylindrical Coordinates: Let the equation of the surface be $z = f(r, \phi)$. Replacing $dx\,dy$ by $r\,dr\,d\phi$,

We get $V = \iint_R z\,r\,dr\,d\phi$.

Q. (1) Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Soln: The two curves intersect at points whose abscissae are given by $4x - x^2 = x$.

$x^2 - 3x = 0$. i.e. $x = 0, 3$

$\therefore$ the required area lies between $x = 0, x = 3$ and $y = x$, $y = 4x - x^2$.

$\therefore$ Required area $= \int_0^3 \int_x^{4x-x^2} dy\,dx = \int_0^3 [y]_x^{4x-x^2}$

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$\therefore$ Required area $= \int_0^3 \int_x^{4x-x^2} dy\,dx = \int_0^3 [y]_x^{4x-x^2} dx$

$= \int_0^3 (4x - x^2 - x)\,dx = \int_0^3 (3x - x^2)\,dx$.

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = 4.5$. Ans

Q (2) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y + z = 4$ and $z = 0$.

Soln: We have $z = 4 - y$ is to be integrated over the circle $x^2+y^2 = 4$, $x$ varies from $0$ to $\sqrt{4-y^2}$ and $y$ varies from $-2$ to $2$.

$\therefore$ Required volume $= 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} z\,dx\,dy = 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} (4-y)\,dx\,dy$

$= 2\int_{-2}^{2} (4-y)\,[x]_0^{\sqrt{4-y^2}}\,dy = 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$

$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$

[illegible]

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$$V = 8\int_{-2}^{2}\sqrt{4-y^2}\,dy$$ {The Second term vanishes as the integrand is an odd fn.}

$$= 8\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2} = 16\pi.$$

Ans